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Satake superdiagrams and Iwasawa decomposition of some hyperbolic Kac–Moody superalgebras

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Abstract

The involutive automorphisms of hyperbolic Kac–Moody superalgebras are computed from the Satake superdiagrams corresponding to these algebras. These are then used to furnish a general treatment of the Iwasawa decomposition of these algebras. In particular, we consider $\hat{\mathrm{D}}^{(1)}(2, 1, \alpha)$ and $\hat{\mathrm{A}}^{(1)}(0, 1)$ as representative examples for the purpose of illustration.

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1. Introduction

The Dynkin diagrams of hyperbolic Kac–Moody superalgebras [1–3] can be constructed by drawing all possible Lie or affine diagrams and then adding one extra root with all possible lengths. Then we try connecting new roots to the old one in all possible ways consistent with a symmetrizable Cartan matrix. The resulting diagram can be tested by removing any vertex to see whether it reduces to simple or affine superalgebras (semisimple algebras are also allowed). A diagram that survives the above operation is of hyperbolic type. Once the Dynkin diagrams are known, one can construct their corresponding Satake superdiagrams. Previously, Satake superdiagrams have been used to obtain all the real forms of simple Lie superalgebras [9]. In a series of papers, we have already shown how this technique can be used to obtain the real forms [9], Iwasawa decomposition [4, 8, 9] of Lie superalgebras, Kac–Moody algebras [10–12] and affine [13] Kac–Moody superalgebras. In continuation of these, we conclude such type of studies with hyperbolic Kac–Moody superalgebras.

The outline for the construction of the Satake superdiagrams of the hyperbolic Kac–Moody superalgebras are briefly delineated in section 2. All possible Satake superdiagrams of $\hat{\mathrm{D}}^{(1)}(2, 1, \alpha)$ and $\hat{\mathrm{A}}^{(1)}(0, 1)$ alongwith their root automorphisms are computed in detail and listed in tables 1 and 2, respectively. Section 3 is devoted to a quick résumé of the procedure for obtaining the Iwasawa decomposition within the present context. The Satake superdiagrams can be constructed for all types of hyperbolic Kac–Moody superalgebras. However, for the Iwasawa decomposition, we restrict ourselves to study only those types of hyperbolic Kac–Moody superalgebras which are extensions of affine algebras by a basic representation, since the root system of this type of algebra is easy to handle. The explicit construction of this decomposition is carried out for the hyperbolic Kac–Moody superalgebras $\hat{\mathrm{D}}^{(1)}(2, 1, \alpha)$ and $\hat{\mathrm{A}}^{(1)}(0, 1)$ in the same section. Section 4 contains a few concluding remarks.

2. Hyperbolic Kac–Moody superalgebras and Satake superdiagrams

We commence this section by taking a look at the hyperbolic Kac–Moody superalgebra $\hat{G}$ which is an extension of the affine superalgebra G by a basic representation and the construction of Satake superdiagrams. Let an affine superalgebra of rank r be characterized by a Cartan matrix $(a_{ij}), ij = 0, 1, \ldots, r$ and a subset $\tau \subset I \equiv \{0, 1, \ldots, r\}$ that identifies the odd generators $e_{\alpha_i}, e_{-\alpha_i}, h_{\alpha_i}$ as generators associated with the simple roots α_i. By convention the root α_0 is the extended root of the affine superalgebra G and the remaining roots α_j, $j = 1, \ldots, r$, are those of the associated finite simple Lie superalgebra. We can extend G by a derivation d which commutes with Cartan generators h_{α_i} and $\left[d, e_{\alpha_i}\right] = -\delta_{i,0} e_{\alpha_i}, \left[d, e_{-\alpha_i}\right] = \delta_{i,0} e_{-\alpha_i}$. The Cartan matrix $\hat{A}$ of the extension of G is defined by $\hat{a}_{ij} = a_{ij}$ for $ij = 0, 1, \ldots, r$, $\hat{a}_{i,-1} = \hat{a}_{-1,i} = -\delta_{i,0}$. The algebra $\hat{G}$ is thus of rank $(r + 1)$. The root α_{-1} is usually referred to as the overextended root. The superalgebra $\hat{G}$ can be defined as the algebra generated by the elements $\left(e_{\alpha_i}, e_{-\alpha_i}, h_{\alpha_i}, i = -1, 0, 1, \ldots, r\right)$ with the relations

$$\begin{aligned} &\left[h_{\alpha_i}, h_{\alpha_j}\right] = 0 \qquad \left[e_{\alpha_i}, e_{-\alpha_j}\right] = \delta_{ij} h_{\alpha_i} \qquad \left[h_{\alpha_i}, e_{\pm\alpha_j}\right] = \pm a_{ji} e_{\pm\alpha_j} \\ &\left(ade_{\alpha_i}\right)^{1-\tilde{a}_{ji}} e_{\alpha_j} = \left(ade_{-\alpha_i}\right)^{1-\tilde{a}_{ji}} e_{-\alpha_j} = 0 \qquad i \neq j \end{aligned} \tag{2.1}$$

where $[\,,]$ stands for the graded product defined by $[x, y] = -(-)^{\deg x \deg y}[y, x]$ with $\deg h_{\alpha_i} = 0, \deg e_{\alpha_i} = \deg e_{-\alpha_i} = 0, i \notin \tau; \deg e_{\alpha_i} = \deg e_{-\alpha_i} = 1, i \in \tau$, and the matrix $\tilde{A} = (\tilde{a}_{ij})$ is deduced from the Cartan matrix $\hat{A} = (\hat{a}_{ij})$ by substituting -1 for the strictly positive elements in the rows with 0 on the diagonal entry and setting $\tilde{a}_{ij} = 2\hat{a}_{ij}$ if $\hat{a}_{ii} = 1$. However, in the case of superalgebra the description given by the above Serre's relation leads in general to a bigger superalgebra than the superalgebra under consideration. So it is necessary to write supplementary relations involving more than two generators, in order to quotient the bigger superalgebra and recover the original one. These supplementary [14] conditions appear when one deals with odd roots of zero length (i.e. $a_{ii} = 0$). The supplementary conditions depend on different kinds of vertices which appear in the Dynkin diagrams. For example, in the case of $A(m, n)$, if α_i is an odd simple root then the supplementary condition,

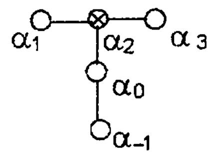
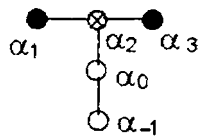
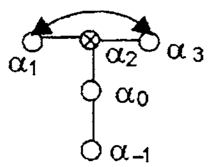
$$\left[\left[\left[e_{\alpha_{i-1}}, e_{\alpha_i}\right], e_{\alpha_{i+1}}\right], e_{\alpha_i}\right] = 0 \tag{2.1a}$$

is necessary. Similarly, different types of relations hold good for different types of vertices, the details of which can be found in [14, 15].

We can now associate with each simple root system a Dynkin diagram. The Satake superdiagrams [9, 13] from the Dynkin diagrams of hyperbolic Kac–Moody superalgebras are achieved with the help of the following prescriptions.

Let R be a root system of hyperbolic Kac–Moody superalgebra. For $\alpha \in R$, let $\bar{\alpha} = (-1)^{|\alpha|}\alpha - \sigma(\alpha)$, where $|\alpha|$ is the degree and $\sigma(\alpha)$ is the image of the root α under the automorphism σ. Let us introduce $R_- = \{\bar{\alpha} \mid \bar{\alpha} \neq 0, \alpha \in R\}$ and $R_0 = \{\alpha \in R \mid \bar{\alpha} = 0\}$. If B_- denotes the basis of R_- and B_0, a basis of R_0, then $B_0 = B \cap R_0$. If $B_- = B \backslash B_0 = \{\alpha_i\}$

Table 1. Satake superdiagrams and involutive automorphisms of $\hat{\mathrm{D}}^{(1)}(2, 1, \alpha)$.

Satake superdiagrams	Involutive automorphisms
(i) 	$-\sigma(\alpha_1) = \alpha_1, -\sigma(\alpha_2) = \alpha_2, -\sigma(\alpha_3) = \alpha_3,$ $-\sigma(\alpha_0) = \alpha_0, -\sigma(\alpha_{-1}) = \alpha_{-1}$
(ii) 	$\sigma(\alpha_1) = \alpha_1, -\sigma(\alpha_2) = \alpha_1 + \alpha_2 + \alpha_3,$ $\sigma(\alpha_3) = \alpha_3, -\sigma(\alpha_0) = \alpha_0, -\sigma(\alpha_{-1}) = \alpha_{-1}$
(iii) 	$-\sigma(\alpha_1) = \alpha_3, -\sigma(\alpha_2) = \alpha_2, -\sigma(\alpha_3) = \alpha_1,$ $-\sigma(\alpha_0) = \alpha_0, -\sigma(\alpha_{-1}) = \alpha_{-1}$

and $B_0 = (\beta_l\}$, then it can be shown that

$$-\sigma(\alpha_i) = \alpha_{\pi(i)} + (-1)^{|\alpha_i|} \sum_l \eta_{il} \beta_l \tag{2.2}$$

where π is the involutive permutation of $\{-1, 0, 1, 2, \ldots, r\}$ and $(-1)^{|\alpha_i|}$, η_{il} are non-negative integers. We should note that $\sigma(\beta_l) = (-1)^{|\beta_l|}\beta_l$ and $\alpha + \sigma(\alpha) \notin R\ \forall\alpha \in R$. We can now associate with B its Satake superdiagrams. In the Dynkin diagrams of B, denote the root α_i by the usual white, grey and black dots and root β_l by black dots. We should note that this black dot is different from the black dot associated with a non-degenerate odd root such as in $B(0, n)$ for instance. If $\pi(i) = k$, then it will be indicated by a double-headed arrow $\leftrightarrow$. We avoid blackening of grey dots, otherwise the uniqueness of Satake superdiagrams would be lost.

As an illustration we restrict ourselves to two examples, $\hat{\mathrm{D}}^{(1)}(2, 1, \alpha)$ and $\hat{\mathrm{A}}^{(1)}(0, 1)$, to highlight the salient features of the scheme.

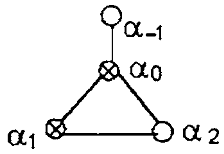
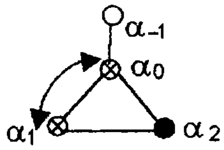
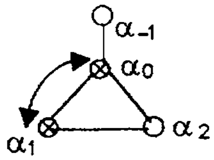
2.1. Satake superdiagrams of $\hat{\mathrm{D}}^{(1)}(2, 1, \alpha)$

The Cartan matrix of $\hat{\mathrm{D}}^{(1)}(2, 1, \alpha)$ is

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 2 & -1 & 0 \\ 0 & -1-\alpha & 1 & 0 & \alpha \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

The five simple roots of $\hat{\mathrm{D}}^{(1)}(2, 1, \alpha)$ are $\alpha_{-1}, \alpha_0, \alpha_1, \alpha_2, \alpha_3$. The possible Satake superdiagrams of $\hat{\mathrm{D}}^{(1)}(2, 1, \alpha)$ along with their root automorphisms are depicted in table 1.

Table 2. Satake superdiagrams and involutive automorphisms of $\hat{A}^{(1)}(0, 1)$.

Satake superdiagrams	Involutive automorphisms
(i) 	$-\sigma(\alpha_1) = \alpha_1, -\sigma(\alpha_2) = \alpha_2,$ $-\sigma(\alpha_0) = \alpha_0, -\sigma(\alpha_{-1}) = \alpha_{-1}$
(ii) 	$-\sigma(\alpha_1) = \alpha_0 + \alpha_2, \sigma(\alpha_2) = \alpha_2,$ $-\sigma(\alpha_0) = \alpha_1 + \alpha_2, -\sigma(\alpha_{-1}) = \alpha_{-1}$
(iii) 	$-\sigma(\alpha_1) = \alpha_0, -\sigma(\alpha_2) = \alpha_2,$ $-\sigma(\alpha_0) = \alpha_1, -\sigma(\alpha_{-1}) = \alpha_{-1}$

2.2. Satake superdiagrams of $\hat{A}^{(1)}(0, 1)$

The Cartan matrix of $\hat{A}^{(1)}(0, 1)$ is

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$$

and the four simple roots of $\hat{A}^{(1)}(0, 1)$ are $\alpha_{-1}, \alpha_0, \alpha_1$ and α_2, where $\alpha_0 = \delta - (\alpha_1 + \alpha_2)$. The possible Satake superdiagrams of $\hat{A}^{(1)}(0, 1)$ along with their root automorphisms can be read from table 2.

3. Iwasawa decomposition of hyperbolic Kac–Moody superalgebra

The notion of direct determination of the Iwasawa decomposition of Lie algebras is now extended to the case of hyperbolic Kac–Moody superalgebras. Let $\hat{G}_c$ be a real hyperbolic Kac–Moody superalgebra generated from its compact real form $\hat{G}_K$ by an involutive automorphism defined with respect to the Cartan subalgebra H of $\hat{G}_c$, which is the complexification of $\hat{G}$.

The following commutation relations are satisfied by the elements of $\hat{G}_c$:

$$\begin{aligned} &[h, e_\alpha] = \alpha(h) e_\alpha && h \in H \quad \alpha \in \Delta \\ &[e_\alpha, e_\beta] = \begin{cases} N_{\alpha\beta} e_{\alpha+\beta} & \text{if } \alpha + \beta \text{ is a root} \\ 0 & \text{otherwise} \end{cases} \\ &[e_\alpha, e_{-\alpha}] = h_\alpha && h_\alpha \in H. \end{aligned} \tag{3.1}$$

Here Δ denotes the set of roots of $\hat{G}_c$ with respect to H and the Killing form is defined by $B(e_\alpha, e_{-\alpha}) = -1$. Here $\alpha(h) = B(h, h_\alpha)$. The compact real form $\hat{G}_K$ may be taken to consist of

$$\mathrm{i}h_\alpha, \quad \alpha = \alpha_{-1}, \alpha_0, \alpha_1, \ldots, \alpha_r \quad \text{and} \quad (e_\alpha + e_{-\alpha}),\ \mathrm{i}(e_\alpha - e_{-\alpha}) \quad \forall\alpha \in \Delta.$$

Let K be the maximal compact subalgebra of $\hat{G}$ defined in such a way that $a \in K$ if $a \in \hat{G}$ and $\sigma a = a$, and let P be the subspace such that $a \in P$ if $a \in \hat{G}$ and $\sigma a = -a$. Thus K and P are given by

$$\begin{aligned} K &= \{\mathrm{i}h_\alpha, \text{for}\, \alpha = \alpha_{-1}, \alpha_0, \alpha_1, \ldots, \alpha_r \text{ and } (e_\alpha + e_{-\alpha}), \mathrm{i}(e_\alpha - e_{-\alpha})\, \forall\alpha | \exp\alpha(h) = +1\} \\ P &= \{\mathrm{i}(e_\alpha + e_{-\alpha}), (e_\alpha - e_{-\alpha})\, \forall\alpha | \exp\alpha(h) = -1\}. \end{aligned} \tag{3.2}$$

Let A be the maximal Abelian subalgebra of P with dimension $|m|$ and M be the centralizer of A in K. Thus, A may be taken to have a basis consisting of the elements of the form $\mathrm{i}(e_\alpha + e_{-\alpha})$. Let R_A denote the set of positive roots that appear in this way in A. Similarly, M may be taken to have a basis consisting of the elements of the form $(e_\alpha + e_{-\alpha})$, with a set of positive roots α appearing this way in M, being denoted by R_M, together possibly with some elements of $H \cap \hat{G}$. If $h'' \in H \cap \hat{G}$ is a member of M, then $\alpha(h'') = 0\ \forall\alpha \in R_A \cup R_M$. Complexification of $A \oplus M$ together with the derivation d' gives a Cartan subalgebra H' of $\hat{G}_c$ with the basis $H'_{-1}, H'_0, H'_1, \ldots, d'$. Now there exists an inner automorphism $V : H' \to H$, i.e. $H_j = VH'_j$, where $V = \prod_\alpha V_\alpha, \alpha \in \Delta$.

Let Δ^+ be the set of positive roots, then

$$h_\alpha = \sum_{j=-1} b_j(\alpha) H_j.$$

Thus $\alpha \in \Delta^+$ iff $b_j(\alpha) > 0$, where j is the least index such that $b_j(\alpha) \neq 0$. The positive roots can again be divided into the following classes:

$$\begin{aligned} &\text{(i) } \Delta^+_+ = \{\alpha | \alpha \in \Delta^+, \alpha(h) \neq \alpha(V\sigma V^{-1}h)\, \forall h \in h\} \\ &\text{(ii) } \Delta^+_- = \{\alpha | \alpha \in \Delta^+, \alpha(h) = \alpha(V\sigma V^{-1}h)\, \forall h \in h\}. \end{aligned} \tag{3.3}$$

Let the subalgebra $\tilde{N}$ be spanned by the elements $V^{-1}e_\alpha$ for $\alpha \in \Delta^+_+$ and $N = \tilde{N} \cap \hat{G}$, where $\tilde{N}$ and N are the nilpotent subalgebras of $\hat{G}_c$ and $\hat{G}$, respectively. Thus the Iwasawa decomposition of $\hat{G}$ is given by

$$\hat{G} = K \oplus A \oplus N. \tag{3.4}$$

3.1. Iwasawa decomposition of $\hat{\mathrm{D}}^{(1)}(2, 1, \alpha)$

Let us consider the involutive automorphism of $\hat{\mathrm{D}}^{(1)}(2, 1, \alpha)$ determined by any one of the Satake superdiagrams, say (ii) of table 1. The simple root automorphism is given by

$$\begin{aligned} &-\sigma(\alpha_{-1}) = \alpha_{-1} && -\sigma(\alpha_0) = \alpha_0 && \sigma(\alpha_1) = \alpha_1 \\ &-\sigma(\alpha_2) = \alpha_1 + \alpha_2 + \alpha_3 && \sigma(\alpha_3) = \alpha_3. && \end{aligned} \tag{3.5}$$

The positive roots of $\hat{\mathrm{D}}^{(1)}(2, 1, \alpha)$ are given by

$$\begin{aligned} \Delta = \{&n_1\alpha_{-1} \pm \alpha_1 + n_2\delta, n_1\alpha_{-1} \pm \alpha_2 + n_2\delta, n_1\alpha_{-1} \pm \alpha_3 + n_2\delta, n_1\alpha_{-1} \pm (\alpha_1 + \alpha_2) + n_2\delta, \\ &n_1\alpha_{-1} \pm (\alpha_2 + \alpha_3) + n_2\delta, n_1\alpha_{-1} \pm (\alpha_1 + \alpha_2 + \alpha_3) + n_2\delta, \\ &n_1\alpha_{-1} \pm (\alpha_1 + 2\alpha_2 + \alpha_3) + n_2\delta, n_1\alpha_{-1} + n_2\delta \text{ where } n_1 = 0, 1, 2, \ldots, n_2 \\ &\quad \text{and } n_2 \in Z^+\}. \end{aligned} \tag{3.6}$$

We can now apply simple root automorphisms to find the automorphisms of other roots and see that these can be separated into two categories, i.e.

$$\exp\alpha(h) = +1 \qquad \text{for } \{n_1\alpha_{-1} + \alpha_1 + n_2\delta, n_1\alpha_{-1} + \alpha_2 + n_2\delta, n_1\alpha_{-1} + \alpha_3 + n_2\delta, n_1\alpha_{-1} + (\alpha_1 + \alpha_2) + n_2\delta,\ n_1\alpha_{-1} - (\alpha_2 + \alpha_3) + n_2\delta, n_1\alpha_{-1} - (\alpha_1 + \alpha_2 + \alpha_3) + n_2\delta\} \tag{3.7}$$

$$\exp\alpha(h) = -1 \qquad \text{for } \{n_1\alpha_{-1} - \alpha_1 + n_2\delta, n_1\alpha_{-1} - \alpha_2 + n_2\delta, n_1\alpha_{-1} - \alpha_3 + n_2\delta, n_1\alpha_{-1} - (\alpha_1 + \alpha_2) + n_2\delta, n_1\alpha_{-1} + (\alpha_2 + \alpha_3) + n_2\delta, n_1\alpha_{-1} + (\alpha_1 + \alpha_2 + \alpha_3) + n_2\delta, n_1\alpha_{-1} \pm (\alpha_1 + 2\alpha_2 + \alpha_3) + n_2\delta,\ n_1\alpha_{-1} + n_2\delta\}. \tag{3.8}$$

Thus for $\hat{\mathrm{D}}^{(1)}(2, 1, \alpha)$, K and P are given by

$$K = \{\mathrm{i}h_\alpha, \text{for } \alpha = \alpha_{-1}, \alpha_0, \alpha_1, \alpha_2, \alpha_3 \text{ and } (e_\alpha + e_{-\alpha}), \mathrm{i}(e_\alpha - e_{-\alpha}) \text{ for } \alpha \text{ given by (3.7)}\} \tag{3.9}$$

$$P = \{\mathrm{i}(e_\alpha + e_{-\alpha}), (e_\alpha - e_{-\alpha}) \text{ for } \alpha \text{ given by (3.8)}\}. \tag{3.10}$$

We now select a maximal Abelian subalgebra A and see that it is one dimensional and has basis elements

$$H_{-1} = \mathrm{i}(e_\alpha - e_{-\alpha}) \qquad \text{for} \quad \alpha = (\alpha_1 + 2\alpha_2 + \alpha_3) + m\delta \tag{3.11}$$

so, now we have

$$R_A = (\alpha_1 + 2\alpha_2 + \alpha_3) + m\delta. \tag{3.12}$$

Note that M is four dimensional with $R_M = (\alpha_1, \alpha_3)$ and its basis elements are given by

$$\begin{aligned} -\mathrm{i}H'_0 &= \left(e_{\alpha_1} + e_{-\alpha_1}\right) \qquad & -\mathrm{i}H'_1 &= \left(e_{\alpha_3} + e_{-\alpha_3}\right) \\ -\mathrm{i}H'_2 &= \mathrm{i}h_{(m+n)\delta} & -\mathrm{i}H'_3 &= \mathrm{i}\left\{h_{\alpha_{-1}} - mh_{\alpha_2}\right\}. \end{aligned} \tag{3.13}$$

H'_{-1}, H'_0, H'_1, H'_2 and H'_3 together with the scaling element d' are the elements of the Cartan subalgebra H'. The required inner automorphism V that maps H' onto H is then given by

$$V = \prod_\alpha V_\alpha \qquad \forall\alpha \in R_A \cup R_M \tag{3.14}$$

where

$$V_\alpha = \exp\{ad\{\mathrm{i}a_\alpha(e_\alpha - e_{-\alpha})\}\} \qquad \text{and} \qquad a_\alpha = \frac{\pi}{[8(\alpha, \alpha)]^{1/2}}. \tag{3.15}$$

In our case,

$$V = V_{\alpha_1+2\alpha_2+\alpha_3+m\delta} V_{\alpha_1} V_{\alpha_3}. \tag{3.16}$$

Applying this to the Cartan subalgebra of H', we get

$$\begin{aligned} H_{-1} &= -\left(\frac{1}{1+\alpha}\right)^{1/2} h_{\alpha_1+2\alpha_2+\alpha_3+(m+n)\delta} \\ H_0 &= -h_{\alpha_1+(m+n)\delta} \\ H_1 &= -\left(\frac{1}{\alpha}\right)^{1/2} h_{\alpha_3+(m+n)\delta} \\ H_2 &= -h_{(m+n)\delta} \\ H_3 &= -h_{\alpha_{-1}} + mh_{\alpha_2}. \end{aligned} \tag{3.17}$$

With respect to this Cartan subalgebra, the set of positive roots is given by

$$\begin{aligned}
\Delta^+ = \{&-(n_1\alpha_{-1} + \alpha_1 + n_2\delta), -(n_1\alpha_{-1} - \alpha_1 + n_2\delta) \text{ for } mn_1 \neq 0,\\
&(n_1\alpha_{-1} - \alpha_1 + n_2\delta) \text{ for } mn_1 = 0,\\
&-(n_1\alpha_{-1} + \alpha_2 + n_2\delta), (n_1\alpha_{-1} - \alpha_2 + n_2\delta) \text{ for } mn_1 = 0,\\
&-(n_1\alpha_{-1} - \alpha_2 + n_2\delta) \text{ for } mn_1 \neq 0, -(n_1\alpha_{-1} + \alpha_3 + n_2\delta),\\
&-(n_1\alpha_{-1} + (\alpha_1 + \alpha_2) + n_2\delta), -(n_1\alpha_{-1} - \alpha_3 + n_2\delta) \text{ for } mn_1 \neq 0,\\
&(n_1\alpha_{-1} - \alpha_3 + n_2\delta) \text{ for } mn_1 = 0,\\
&(n_1\alpha_{-1} - (\alpha_1 + \alpha_2) + n_2\delta) \text{ for } mn_1 = 0, 1,\\
&-(n_1\alpha_{-1} - (\alpha_1 + \alpha_2) + n_2\delta) \text{ for } mn_1 \neq 0, 1,\\
&-(n_1\alpha_{-1} - (\alpha_2 + \alpha_3) + n_2\delta) \text{ for } mn_1 \neq 0, 1,\\
&(n_1\alpha_{-1} - (\alpha_2 + \alpha_3) + n_2\delta) \text{ for } mn_1 = 0, 1,\\
&-(n_1\alpha_{-1} - (\alpha_2 + \alpha_3) + n_2\delta),\\
&(n_1\alpha_{-1} - (\alpha_1 + \alpha_2 + \alpha_3) + n_2\delta) \text{ for } mn_1 = 0, 1,\\
&-(n_1\alpha_{-1} - (\alpha_1 + \alpha_2 + \alpha_3) + n_2\delta) \text{ for } mn_1 \neq 0, 1,\\
&-(n_1\alpha_{-1} + (\alpha_1 + \alpha_2 + \alpha_3) + n_2\delta),\\
&-(n_1\alpha_{-1} + (\alpha_1 + 2\alpha_2 + \alpha_3) + n_2\delta),\\
&(n_1\alpha_{-1} - (\alpha_1 + 2\alpha_2 + \alpha_3) + n_2\delta) \text{ for } mn_1 = 0, 1, 2,\\
&-(n_1\alpha_{-1} - (\alpha_1 + 2\alpha_2 + \alpha_3) + n_2\delta) \text{ for } mn_1 \neq 0, 1, 2,\\
&(n_1\alpha_{-1} + n_2\delta) \text{ for } mn_1 = 0,\\
&-(n_1\alpha_{-1} + n_2\delta) \text{ for } mn_1 \neq 0\}. \qquad (3.18)
\end{aligned}$$

Now it can be divided into two categories:

$$\begin{aligned}
\Delta^+_- = \{&-(n_1\alpha_{-1} + \alpha_1 + n_2\delta) \text{ for } mn_1(1+\alpha) = 4,\\
&-(n_1\alpha_{-1} + \alpha_3 + n_2\delta) \text{ for } mn_1\alpha = 4,\\
&-(n_1\alpha_{-1} + (\alpha_1 + \alpha_2) + n_2\delta) \text{ for } mn_1(1+\alpha) = 2,\\
&-(n_1\alpha_{-1} + (\alpha_2 + \alpha_3) + n_2\delta) \text{ for } mn_1(1+\alpha) = 2\alpha,\\
&-(n_1\alpha_{-1} + (\alpha_1 + \alpha_2 + \alpha_3) + n_2\delta) \text{ for } mn_1(1+\alpha) = (4 + 2\alpha),\\
&-(n_1\alpha_{-1} + (\alpha_1 + \alpha_2 + \alpha_3) + n_2\delta) \text{ for } mn_1 = 2\} \qquad (3.19)
\end{aligned}$$

and

$$\begin{aligned}
\Delta^+_+ = \{&-(n_1\alpha_{-1} + \alpha_1 + n_2\delta) \text{ for } mn_1(1+\alpha) \neq 4,\\
&-(n_1\alpha_{-1} - \alpha_1 + n_2\delta), -(n_1\alpha_{-1} \pm \alpha_2 + n_2\delta), -(n_1\alpha_{-1} - \alpha_3 + n_2\delta),\\
&-(n_1\alpha_{-1} + \alpha_3 + n_2\delta) \text{ for } mn_1\alpha \neq 4,\\
&-(n_1\alpha_{-1} + (\alpha_1 + \alpha_2) + n_2\delta) \text{ for } mn_1(1+\alpha) \neq 2,\\
&-n_1\alpha_{-1} - (\alpha_1 + \alpha_2) + n_2\delta, -(n_1\alpha_{-1} - (\alpha_2 + \alpha_3) + n_2\delta),\\
&-(n_1\alpha_{-1} - (\alpha_1 + \alpha_2 + \alpha_3) + n_2\delta),\\
&-(n_1\alpha_{-1} - (\alpha_1 + 2\alpha_2 + \alpha_3) + n_2\delta), -(n_1\alpha_{-1} + n_2\delta)\}. \qquad (3.20)
\end{aligned}$$

For $\hat{\mathrm{D}}^{(1)}(2, 1, \alpha)$, the basis elements of N are given by the structure

$$V^{-1}e_\alpha \qquad \text{where} \quad \alpha \in \Delta^+_+. \qquad (3.21)$$

This structure can be calculated explicitly by applying the properties of inner automorphism. For example, the element

$$V^{-1}e_{-n_1\alpha_{-1}-\alpha_1-n_2\delta} = \frac{1}{\sqrt{2}}e_{-(n_1\alpha_{-1}+\alpha_1+n_2\delta)} - \frac{\mathrm{i}}{[2(1+t^{2m})]^{1/2}}e_{-n_1\alpha_{-1}-n_2\delta} - \frac{\mathrm{i}}{[2(1+t^{2m})]^{1/2}}Ae_{-[n_1\alpha_{-1}+\alpha_1+2\alpha_2+\alpha_3+(m+n)\delta]} \tag{3.22}$$

where

$$A = \mathrm{sgn}\big(N_{(\alpha_1+2\alpha_2+\alpha_3+m\delta)-(n_1\alpha_{-1}+\alpha_1+2\alpha_2+\alpha_3+(m+n)\delta)}\big). \tag{3.23}$$

The elements of N can be known by considering the element $N \cap \hat{G}$. The required Iwasawa decomposition is

$$\hat{\mathrm{D}}^{(1)}(2, 1, \alpha) = K \oplus A \oplus N. \tag{3.24}$$

There are $2^{|m|}$ classes of parabolic subalgebra, where m is the dimension of A. In this case $|m| = 1$, so there will be two parabolic subalgebras, one being minimal parabolic and the other is the algebra itself. The minimal parabolic subalgebra is given by

$$P = M \oplus A \oplus N \tag{3.25}$$

where

$$\begin{aligned} M &= \{(e_\alpha + e_{-\alpha}) \text{ for } \alpha = \alpha_1 \text{ and } \alpha_3\} \\ A &= \{\mathrm{i}(e_\alpha + e_{-\alpha}) \text{ for } \alpha = \alpha_1 + 2\alpha_2 + \alpha_3 + m\delta\}. \end{aligned} \tag{3.26}$$

3.2. *Iwasawa decomposition of* $\hat{\mathrm{A}}^{(1)}(0, 1)$

Here, we consider the involutive root automorphism of $\hat{\mathrm{A}}^{(1)}(0, 1)$ determined by the Satake diagram (ii) of table 2. The root automorphisms are given by

$$\begin{aligned} &-\sigma(\alpha_1) = \alpha_0 + \alpha_2 \qquad &-\sigma(\alpha_0) = \alpha_1 + \alpha_2 \\ &\sigma(\alpha_2) = \alpha_2 \qquad &-\sigma(\alpha_{-1}) = \alpha_{-1}. \end{aligned} \tag{3.27}$$

The above equation can be rewritten as

$$\begin{aligned} &-\sigma(\alpha_1) = -\alpha_1 + \delta \qquad &\sigma(\alpha_2) = \alpha_2 \\ &-\sigma(\delta - \alpha_1 - \alpha_2) = \alpha_1 + \alpha_2 \qquad &-\sigma(\alpha_{-1}) = \alpha_{-1}. \end{aligned} \tag{3.28}$$

The positive roots are given by

$$\Delta = \{n_1\alpha_{-1} \pm \alpha_1 + n_2\delta, n_1\alpha_{-1} \pm \alpha_2 + n_2\delta, n_1\alpha_{-1} \pm (\alpha_1 + \alpha_2) + n_2\delta, n_1\alpha_{-1} + n_2\delta, \text{ where } n_1 = 0, 1, \ldots, n_2 \text{ and } n_2 \in Z^+\}. \tag{3.29}$$

After applying a simple root automorphism, we divide the positive root system into two categories with

$$\exp\alpha(h) = +1 \qquad \text{for } \{n_1\alpha_{-1} + \alpha_1 + n_2\delta, n_1\alpha_{-1} + \alpha_2 + n_2\delta, n_1\alpha_{-1} + \alpha_1 + \alpha_2 + n_2\delta\} \tag{3.30}$$

$$\exp\alpha(h) = -1 \qquad \text{for } \{n_1\alpha_{-1} - \alpha_1 + n_2\delta, n_1\alpha_{-1} - \alpha_2 + n_2\delta, n_1\alpha_{-1} - \alpha_1 - \alpha_2 + n_2\delta, n_1\alpha_{-1} + n_2\delta\}. \tag{3.31}$$

Thus for $\hat{\mathrm{A}}^{(1)}(0, 1)$, K and P are given by

$$K = \{\mathrm{i}h_\alpha \text{ for } \alpha = \alpha_{-1}, \alpha_0, \alpha_1, \alpha_2 \text{ and } (e_\alpha + e_{-\alpha}), \mathrm{i}(e_\alpha - e_{-\alpha}) \text{ for } \alpha \text{ given by equation (3.30)}\}. \tag{3.32}$$

$$P = \{\mathrm{i}(e_\alpha + e_{-\alpha}), (e_\alpha + e_{-\alpha}) \text{ for } \alpha \text{ given by equation (3.31)}\}. \tag{3.33}$$

We now select a maximal Abelian subalgebra A and see that A is one dimensional and its basis elements are given by

$$H'_{-1} = \mathrm{i}(e_\alpha + e_{-\alpha}) \qquad \text{for} \quad \alpha = -\alpha_2 + m\delta \quad m \in Z^+. \tag{3.34}$$

So we have $R_A = (-\alpha_2 + m\delta)$ and R_M is empty, m is three dimensional and its basis elements are given by

$$\begin{aligned} -\mathrm{i}H'_0 &= \mathrm{i}\left[h_{\alpha_2+(m+n)\delta} + 2h_{\alpha_1+(m+n)\delta}\right] \\ -\mathrm{i}H'_1 &= \mathrm{i}h_{(m+n)\delta} \qquad -\mathrm{i}H'_2 = \mathrm{i}\left(h_{\alpha_{-1}} + mh_{\alpha_2}\right). \end{aligned} \tag{3.35}$$

The elements $H'_{-1}, H'_0, H'_1, H'_2$ together with the scaling elements d' form the Cartan subalgebra H'. Defining the inner automorphism of $\hat{\mathrm{A}}^{(1)}(0,1)$ as

$$V = V_{-\alpha_2+n\delta} = \exp\left[ad\left[a_{-\alpha_2+n\delta}\left(e_{-\alpha_2+n\delta} - e_{\alpha_2-n\delta}\right)\right]\right] \tag{3.36}$$

where

$$a_{-\alpha_2+n\delta} = \frac{\prod}{t^n[8(\alpha_2,\alpha_2)]^{1/2}} \tag{3.37}$$

and applying to the Cartan subalgebra H', we obtain

$$\begin{aligned} H_{-1} &= 2^{1/2}h_{-\alpha_2+(m+n)\delta} \\ H_0 &= -\left(h_{\alpha_2+(m+n)\delta} + 2h_{\alpha_1+(m+n)\delta}\right) \\ H_1 &= -h_{(m+n)\delta} \\ H_2 &= -h_{\alpha_{-1}} - \frac{m}{2}h_{\alpha_2}. \end{aligned} \tag{3.38}$$

The set of positive roots with respect to the Cartan subalgebra is given by

$$\begin{aligned} \Delta^+ = \{&(n_1\alpha_{-1} + \alpha_1 + (m+n)\delta), (n_1\alpha_{-1} - \alpha_1 + (m+n)\delta) \text{ for } mn_1 \neq 0, \\ &-(n_1\alpha_{-1} - \alpha_1 + (m+n)\delta) \text{ for } mn_1 = 0, \\ &(n_1\alpha_{-1} + \alpha_2 + (m+n)\delta) \text{ for } mn_1 \geqslant 2, \\ &-(n_1\alpha_{-1} + \alpha_2 + (m+n)\delta) \text{ for } mn_1 = 0, 1, \\ &(n_1\alpha_{-1} - \alpha_2 + (m+n)\delta), \\ &(n_1\alpha_{-1} + \alpha_1 + \alpha_2 + (m+n)\delta) \text{ for } mn_1 \geqslant 2, \\ &-(n_1\alpha_{-1} + \alpha_1 + \alpha_2 + (m+n)\delta) \text{ for } mn_1 = 0, 1, \\ &(n_1\alpha_{-1} - \alpha_1 - \alpha_2 + (m+n)\delta), \\ &(n_1\alpha_{-1} + (m+n)\delta) \text{ for } mn_1 \neq 0, \\ &-(n_1\alpha_{-1} + (m+n)\delta) \text{ for } mn_1 = 0\}. \end{aligned} \tag{3.39}$$

The sets Δ^+_+ and Δ^+_- can similarly be written as

$$\begin{aligned} \Delta^+_- = \{&(n_1\alpha_{-1} + \alpha_1 + (m+n)\delta) \text{ for } mn_1 = 1, \\ &(n_1\alpha_{-1} - \alpha_2 + (m+n)\delta) \text{ for } mn_1 = 2, \\ &(n_1\alpha_{-1} - \alpha_1 - \alpha_2 + (m+n)\delta) \text{ for } mn_1 = 2, \\ &(n_1\alpha_{-1} + (m+n)\delta) \text{ for } mn_1 = 0\} \end{aligned} \tag{3.40}$$

and

$$\Delta^+_+ = \frac{\Delta^+}{\Delta^+_-}. \tag{3.41}$$

As before, for $\hat{\mathrm{A}}^{(1)}(0,1)$, the elements of N are given by $V^{-1}e_\alpha$, where $\alpha \in \Delta^+_+$.

These structures can be calculated explicitly by applying the properties of inner automorphisms. For example, the element $V^{-1}e_{n_1\alpha_{-1}+\alpha_1+(m+n)\delta}$ is given by

$$V^{-1}e_{n_1\alpha_{-1}+\alpha_1+(m+n)\delta} = \frac{1}{1+t^{2n}}e_{n_1\alpha_{-1}+\alpha_1+(m+n)\delta} + \frac{1}{1+t^{2n}}\left[\operatorname{sgn} N_{-\alpha_2,n_1\alpha_{-1}+\alpha_1+\alpha_2}\right]e_{n_1\alpha_{-1}+\alpha_1+\alpha_2+(m+2n)\delta}. \quad (3.42)$$

The elements of N can be known by considering the elements $N \cap \hat{G}$ as before. The required Iwasawa decomposition is then written as

$$\hat{A}^{(1)}(0,1) = K \oplus A \oplus N. \quad (3.43)$$

4. Concluding remark

The results presented in this paper owe their genesis to the relative simplicity and elegance of the techniques of Satake superdiagrams corresponding to the particular hyperbolic Kac–Moody superalgebras considered here. The involutive automorphism [5, 6] has been obtained from a modified formula. The treatment is general enough to be applicable to a variety of problems, for example, we saw that these two hyperbolic Kac–Moody superalgebras have three real forms each up to an isomorphism, because there are three non-isomorphic Satake superdiagrams. Thus, determining all the Satake superdiagrams of a algebra, we can classify easily all its real forms. Also, this technique can be used to determine the associated symmetric superspaces [7]. The classification of symmetric superspaces in terms of Satake superdiagrams is currently in progress. Also the Iwasawa decomposition of such algebras readily leads to Langalands decomposition, thereby facilitating the determination of parabolic subalgebras, which are necessary for obtaining the corresponding induced representation, with the help of Schmidt's construction.

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